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# Field and Particles - The Quantized Blowup Mechanism

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## 1 Introduction

The purpose of the present paper is to study blowup mechanism of a system of cross diffusion arising in mathematical biology and statistical mechanics. That is,

$$\left.\begin{aligned} u_t &= \nabla\cdot(\nabla u - u\nabla v) \\ 0 &= \Delta v - av + u \end{aligned}\right\} \quad \text{in} \quad \Omega\times(0,T)$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega\times(0,T)$$
$$u|_{t=0} = u_0(x) \quad \text{on} \quad \Omega, \tag{1}$$

where $\Omega\subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $a>0$ is a constant, and $\nu$ is the outer unit vector on $\partial\Omega$.

In the context of mathematical biology, it was proposed by Nagai [16] as a simplified form of the one given by Keller and Segel [15]. Here, $u=u(x,t)$ and $v=v(x,t)$ stand for the density of cellular slime molds and the concentration of chemical substances secreted by themselves, respectively, at the position $x\in\Omega$ and the time $t>0$.

In this case, the first equation describes the conservation of mass, where the flux of $u$ is given by $\mathcal{F}=-\nabla u+u\nabla v$, as

$$\frac{d}{dt}\int_\omega u = -\int_{\partial\omega}\mathcal{F}\cdot\nu$$

holds for any subdomain $\omega\subset\subset\Omega$. Therefore, the effect of diffusion $-\nabla u$ and that of chemotaxis $u\nabla v$ are competing for $u$ to vary. Sometimes it is replaced by

$$u_t = \nabla\cdot(\nabla A(u) - u\nabla\chi(v)) + f(u,v)$$

to describe realistic spatial patterns such as the *streaming*. This case is referred to as the *generalized system*, where $\chi = \chi(v)$ acts as a *sensitivity function*. Among many other works, Harada, Senba, and Suzuki [9] showed that if $f(u,v) = 0$, $A(u) = au^2 + u$ with $a > 0$, and $\chi(v) = v$, then the solution exists globally in time.

In the original form, the second equation takes

$$\tau v_t = \Delta v - av + u \qquad \text{in} \qquad \Omega \times (0,T)$$

and the initial value of $v$ is also prescribed, where $\tau > 0$ is a constant. In this case, $v$ is subject to the linear diffusion equation, provided with the dissipative term $-av$ and also with the growth term proportional to $u$. Here, $\tau$ comes from the time scale of $v$ relative to $u$ and it is natural to assume $0 < \tau \ll 1$. Putting $\tau = 0$ gives (1).

In the context of statistical mechanics, typically the bounded domain $\Omega$ is replaced by the whole space $\mathbf{R}^n$ and the second equation of (1) takes the form

$$v(x,t) = \int K(x,y)u(y,t)dy, \tag{2}$$

where

$$K(x,y) = \begin{cases} \frac{1}{2}|x-y| & (n=1) \\ \frac{1}{2\pi}\log\frac{1}{|x-y|} & (n=2) \\ \frac{1}{4\pi|x-y|} & (n=3) \end{cases} \tag{3}$$

denotes $(-1)$ times potential deriven by the gravitational force. It is concerned with the motion of mean field of self-interacting particles, and is derived from the Langevin and the Fokker-Planck equations. Therefore, while the first equation of (1) is concerned with the mass conservation of particles, the second one replaced by (2) is the description of the total field of gravitational force made by those particles. See Bavaud [3] and Wolansky [36] for details.

This form of (2) is very close to the second equation of (1), as it is equivalent to

$$v(x,t) = \int_\Omega G(x,y)u(y,t)dy, \tag{4}$$

where $G(x,y)$ denotes the Green's function for $-\Delta_N + a$. In fact, we have

$$G(x,y) = H(x,y) + \begin{cases} K(x,y) & (y \in \Omega) \\ 2K(x,y) & (y \in \partial\Omega) \end{cases} \tag{5}$$

with $H \in C^{1,\theta}((\Omega \times \Omega) \cup (\Omega \times \partial\Omega) \cup (\partial\Omega \times \partial\Omega))$. Namely, the second equation of (1) is regarded as a description of the field created by particles.

In mathematical biology, other forms of the second equation are proposed by Jäger and Luckhaus [14] and Diaz and Nagai [6]. They are described totally as

$$\tau \frac{dv}{dt} + Av = u \qquad \text{in} \qquad L^2(\Omega), \tag{6}$$

where $A > 0$ is a self-adjoint operator with the compact resolvent. Here, $\tau$ is a non-negative constant. We call (1) with the second equation replaced by (6) with $\tau > 0$ the *full system*. There the additional initial condition $v|_{t=0} = v_0(x)$ is imposed. If $\tau = 0$, the initial value is only taken for $u$ as in (1). We call this case the *simplified system*. Thus, (1) is regarded as a simplified system of chemotaxis.

As we have seen, the field created by particles is physical in the simplified system. In this context, we may say that in the full system it is formed through a chemical process in biological media. There is a case that the second equation of (1) is replaced by the ordinary differential equation

$$\tau \frac{\partial v}{\partial t} = u.$$

It is derived from the statistical model of cellular automaton, where effect of transmissive action of the control species is restricted to each cell. Therefore, the field is not formed in the classical sense, but let us call it the biological field. We do not discuss that last case, the biological field, here. See Othmer and Stevens [23].

We can summarize that system (1) describes the motion of mean field of particles whose self-interaction is caused by a physical field such as the gravitational force. In the present paper, we study (1) with $n = 2$, although Herrero, Madina, and Velázquez [10], [11] obtained interesting families of blowup solutions for $n = 3$. In this case of $n = 2$, the unique classical solution exists locally in time if the initial value is smooth. The solution becomes positive if the initial value is non-negative and not identically zero. See Yagi [37] and Biler [4].

Let $T_{\max} > 0$ be the supremum of the existence time of the solution. The following theorem is proven by [25], where $\mathcal{M}(\overline{\Omega})$ denotes the set of measures

on $\overline{\Omega}$, $\rightharpoonup$ the $*$-weak convergence there, and

$$m_*(x_0) \equiv \begin{cases} 8\pi & (x_0 \in \Omega) \\ 4\pi & (x_0 \in \partial\Omega)\,. \end{cases}$$

**Theorem 1** *If $T_{\max} < +\infty$, then there exists a finite set $\mathcal{S} \subset \overline{\Omega}$ and a non-negative function $f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$ such that*

$$u(x,t)dx \quad \rightharpoonup \quad \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx \qquad \text{in} \quad \mathcal{M}(\overline{\Omega}) \tag{7}$$

*with*

$$m(x_0) \geq m_*(x_0) \qquad (x_0 \in \mathcal{S})\,. \tag{8}$$

We have $\|u(t)\|_\infty \to +\infty$ as $t \uparrow T_{\max} < +\infty$ and $\mathcal{S}$ is actually the blowup set of $u$. That is, $x_0 \in \mathcal{S}$ if and only if there exist $x_k \to x_0$ and $t_k \uparrow T_{\max}$ such that $u(x_k, t_k) \to +\infty$. Because

$$\|u(t)\|_1 = \|u_0\|_1 \tag{9}$$

holds for $t \in [0, T_{\max})$, we obtain

$$2 \cdot \sharp(\Omega \cap \mathcal{S}) + \sharp(\partial\Omega \cap \mathcal{S}) \leq \|u_0\|_1 / (4\pi) \tag{10}$$

from (7) and (8). Here and henceforth, $\|\cdot\|_p$ denotes the standard $L^p$ norm on $\Omega$ for $p \in [1, \infty]$. In particular, we get the conclusion that $\|u_0\|_1 < 4\pi$ implies $T_{\max} = +\infty$.

The last fact is related to the conjecture of Childress and Percus [5] concerning the threshold in $L^1$ norm of the initial value for the blowup of the solution. There, it was suspected that $\|u_0\|_1 < 8\pi$ implies $T_{\max} = +\infty$, while $T_{\max} < +\infty$ can happen for $\|u_0\|_1 > 8\pi$. However, the result proven mathematically is that $\|u_0\|_1 < 4\pi$ implies $T_{\max} < +\infty$. It was proven independently by Nagai, Senba, and Yoshida [19], Biler [4], and Gajewski and Zacharias [7]. Furthermore, the condition $\|u_0\|_1 < 4\pi$ is sharp for $T_{\max} = +\infty$ to hold, which was proven later by Nagai [17] and Senba and Suzuki [26].

Conjecture of [5] was obtained by semi-analysis, derivation of the stationary problem and numerical study to its bifurcation diagram concerning radially symmetric solutions. On the other hand, mathematical results are

based on a delicate use of the best constant for the Trudinger-Moser inequality. Finally, relation (7) was conjectured by Nanjundiah [21] and is referred to as the formation of *chemotactic collapses*. In fact, each collapse

$$m(x_0)\delta_{x_0}(dx)$$

stands for a spore made from the slime molds in the context of biology.

Our motivation is to explain those two phenomena, threshold and collapses, uniformly from the blowup mechanism. This project was initiated by Nagai, Senba, and Suzuki [18]. Actually, inequality (10) indicates that the phenomenon of threshold in $\|u_0\|_1$ concerning the blowup of the solution is a consequence of the formation of collapses in the blowup process. It also indicates that the boundary blowup forms a half collapse of the one in the inner blowup. This explains exactly the discrepancy between the conjecture and the theorem. Actually, [5] calculated only radially symmetric solutions ! See Senba and Suzuki [24] for detailed studies on stationary solutions.

Now, we can state our problem. In fact, if equality holds in (8), then it means that the formation of spores occures with the normalized mass. We call it the *quantized* blowup mechanism. This case actually holds in the family of blowup solutions constructed by Herrero and Velázquez [12] by the method of matched asymptotic expansion. The general case was suggested by [24] mentioned above.

Up to now, it has been proven that the mass quantization occurs if the solution is continued after the blowup time ([29]) and if the solution blows-up in infinite time ([28]). In this connection, it is worth mentioning that the Fokker-Planck equation admits the weak solution globally in time, provided that the initial value has a finite second moment and is bounded and summable. See Victory, Jr. [34].

Here we note that the Fokker-Planck equation is concerned with the case that the distribution of particles is thin. Therefore, we can suspect that the mass quantization to (1) occurs if the concentration speed is appropriately rapid. Actually, the present paper shows that the mass quantization occurs if the concentration around the blowup point has a parabolic envelop in $(x,t)$ space.

Is any blowup point provided with such a property ? Actually, there is an evidence for this to be. However, more importantly we can get a story for the proof of mass quantization from those considerations. In the last part, we shall describe it and show a theorem obtained actually along that line.

# 2 Physical Backgrounds

Parabolic - elliptic systems of cross diffusion are found in several areas. Here, we mention two of them, semi-conductor device equation and vortex equation derived from the Navier-Stokes equation. The first system is written as

$$\left.\begin{array}{l} n_t = \nabla \cdot (\nabla n - n\nabla\varphi) \\ p_t = \nabla \cdot (\nabla p + p\nabla\varphi) \\ \Delta\varphi = n - p \end{array}\right\} \quad \text{in} \quad \Omega \times (0,T)$$

$$\left.\begin{array}{l} \frac{\partial n}{\partial \nu} - n\frac{\partial \varphi}{\partial \nu} = 0 \\ \frac{\partial p}{\partial \nu} + p\frac{\partial p}{\partial \nu} = 0 \\ \varphi = 0 \end{array}\right\} \quad \text{on} \quad \partial\Omega \times (0,T),$$

where $n = n(x,t)$ and $p = p(x,t)$ are the densities of electron and positron, respectively, and $\varphi = \varphi(x,t)$ is the electric charge field. The case $p = 0$ is easy to handle. Then, we see that the electrons are subject to the self-repulsive force, which makes the system to be dissipative. See Bank [1] for more details.

The second system is given by

$$\left.\begin{array}{l} \omega_t = \nabla \cdot \left(\nabla\omega - \omega\nabla^{\perp}\psi\right) \\ -\Delta\psi = \omega \end{array}\right\} \quad \text{in} \quad \mathbf{R}^2 \times (0,T),$$

where

$$\nabla^{\perp} = \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}$$

for $x = (x_1, x_2)$. It comes from the Navier-Stokes system

$$\left.\begin{array}{r} u_t - \Delta u + u \cdot \nabla u = \nabla p \\ \nabla \cdot u = 0 \end{array}\right\} \quad \text{in} \quad \mathbf{R}^3 \times (0,T),$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{and} \quad \nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}$$

denote the velocity and the gradient operator, respectively. If we take the two dimensional model with $x = (x_1, x_2, 0)$ and $u_3 = 0$, then we get

$$\nabla \times u = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \quad \text{for} \quad \omega = \omega(x_1, x_2).$$

This system is also dissipative but some underlying chaotic features are observed.

Directions of self-interacting forces of those systems, chemotaxis, semiconductor device, and vortices are different, but some common structures are noticed. Let us recall that the second law of thermodynamics; the mean field of many particles is governed by the free energy, decreasing in time. Its local minimum is an equilibrium state, while transient dynamics are controlled by the critical points, especially, non-local minima.

We note that free energy is given by total energy minus entropy. If $\rho = \rho(x) \geq 0$ denotes the density of particles, entropy on the domain $\Omega \subset \mathbf{R}^n$ is given as

$$-\int_\Omega \rho \log \rho.$$

On the other hand, the total energy is composed of kinetic and potential energies so that is given as

$$-\frac{1}{2}\int\int_{\Omega\times\Omega} K(x,y)\rho(x)\rho(y)dxdy + \int_\Omega \rho V,$$

where $-K(x,y)$ and $V(x)$ denote the potentials of self-interactions and external force, respectively. Note that Newton's third law implies

$$K(x,y) = K(y,x).$$

Actually, it is given as (3) if the self-interaction is caused by the gravitational force. Thus, we get a physical question. What is the mean field equation of which free energy is given by

$$\mathcal{F}(\rho) = \int_\Omega \rho \log \rho - \frac{1}{2}\int\int_{\Omega\times\Omega} K(x,y)\rho(x)\rho(y)dxdy + \int_\Omega \rho V \quad ?$$

It has been known that such a system is realized by introducing friction and fluctuations of particles. Actually, we have the work by Bavaud [3] and Wolansky [35], [36].

Recall that the classical theory starts with the Newton equation

$$\frac{dx_i}{dt} = v_i, \qquad m\frac{dv_i}{dt} = \nabla_{x_i}\left\{-mV(x_i) + m^2\sum_{j\neq i} K(x_j,x_i)\right\} \tag{11}$$

for $1 \leq i \leq N$. Letting $N \to \infty$ with $M = mN$ preserved, it asserts the convergence

$$\mu^N(dx,dv,t) = m\sum \delta_{x_i(t)}(dx)\otimes\delta_{v_i(t)}(dv) \rightharpoonup f(x,v,t)dxdv$$

with $f(x,v,t)$ satisfying the kinetic model, referred to as the Jeans-Vlasov equation. In the normal form, it is given as

$$f_t = -\nabla_x \cdot (vf) + \gamma \nabla_v \cdot [f \nabla_x (U+V)]$$
$$U(x,t) = -\int\int K(x,y) f(y,v,t) dv dt$$

In the process of $(dv_i)/(dt) \to 0$, the distribution function $f(x,v,t)$ is replaced by the Maxwellian $\omega(x,t)\pi^{-n/2}e^{-v^2/2}$. If $n=2$, then $\omega(x,t)$ is subject to the vorticity equation derived from the Euler equation, that is,

$$-\Delta\psi = \omega, \qquad \omega_t = -\nabla \cdot \left(\omega \nabla^{\perp} (\psi + V)\right).$$

The stationary state of this equation, $\omega = \omega(x)$ is associated with the elliptic problem

$$-\Delta\psi = g(\psi + V)$$

with the nonlinearity $g$ unknown. This problem was studied by Turkington [32], [33].

If the particles are so concentrated as

$$\omega(x,t) = \sum \delta_{x_j(t)}(dx),$$

then the concetration spots are subject ot the Hamiltonian system

$$\frac{dx_i}{dt} = \nabla^{\perp}_{x_i} \mathcal{H}(x_1, x_2, \cdots, x_N) \qquad (i = 1, 2, \cdots, N), \tag{12}$$

where

$$\mathcal{H}(x_1, x_2, \cdots, x_N) = -\sum_i V(x_i) + \sum_{j \neq i} K(x_i, x_j).$$

If $K(x,y)$ is replaced by $G(x,y)$ in (11), then $\frac{1}{2}\sum_i R(x_i)$ is added to the right-hand side, where $R(x)$ is the regular part of $K(x,y)$ so that $R(x) = H(x,x)$ with $H(x,y)$ defined by (5).

However, the Newton equation is time reversible and this hierarchy of systems is not subject to the second law of thermodynamics, that is, decreasing of the free energy. Actually, this hierarchy is governed by three laws of conservation; mass, momentum, and energy. As a consequence, it has a feature of chaotic motion of particles.

The answer that we know to derive systems provided with the free energy is to replace the Newton equation by the Langevin equation. More precisely,

this requirement is realized when the particles are subject to the friction and random fluctuations:

$$dx_i = v_i dt$$
$$mdv_i = \nabla_{x_i}\left(-mV(x_i) + m^2\sum_{j\neq i} K(x_j, x_i)\right) - \beta v dt + (2\beta kT)^{1/2} dW_t^i$$

Here, $k$, $T$, and $\beta$ are Boltzmann constant, temperature, friction coefficient, respectively, and $(W_t^i)$ denotes the white noise. Its kinetic model, referred to as the Fokker-Planck equation is given as

$$f_t = -\nabla_x \cdot (vf) + \nabla_v \cdot [f\nabla_x (U+V)] + \beta kT\nabla_v \cdot (vf + \Delta_v f)$$
$$U(x,t) = -\int\int K(x,y) f(y,v,t) dy dv,$$

where

$$\rho(x,t) = \int f(x,v,t) dv \qquad \text{and} \qquad \lambda = \int \rho(x,t) dx$$

stand for the density and the total mass, respectively. Then, in the adiabatic limit $\beta \to +\infty$, we have

$$\rho_t = \nabla \cdot (\rho \nabla U) + \nabla \cdot (\rho \nabla V) + \Delta\rho.$$

If $V = 0$ and the kernel $K(x,y)$ is replaced by $G(x,y)$, it is nothing but the simplified system of chemotaxis.

The semi-conductor device equation is obtained similarly by taking the opposite sign of the kernel $G(x,y)$. In those systems of chemotaxis and semi-conductor device the interaction acts attractively and repulsively, respectively. On the other hand, in the vortex equation, the direction of the force that the particles receive is perpendicular to the level lines of the field made by them.

As we shall see, stationary state of the above equation is described by the elliptic problem with the exponential nonlinearity. Furthermore, the localized densities are subject to the gradient flow with $\nabla^\perp$ replaced by $\nabla$ in (12). In this connection, it may be worth noting that the critical point of this $\mathcal{H}(x_1, x_2, \cdots, x_N)$ controls the location of multi-blowup points in the stationary problem. See Nagasaki and Suzuki [20] and Baraket and Pacard [2]. Thus, this hierarchy of equations starts with the free energy as the physical principle. On the other hand, mathematically it is characterized by the quantization of blowup mechanism as we are now describing.

# 3 Mathematical Structures

Several mathematical structures are known to (1) and some of them are valid to the full system. For the moment, we describe them for the full system (1) with the second equation replaced by

$$\tau \frac{\partial v}{\partial t} = \Delta v - av + u$$

but they are valid for the simplified system if the initial value $v_0$ is taken as $(-\Delta_N + a)^{-1} u_0$ and $\tau$ is put to be zero.

First, the positivity of the solution is preserved so that $u_0(x) \geq 0$ and $u_0(x) \not\equiv 0$ imply $u(x,t) > 0$ for $(x,t) \in \overline{\Omega} \times (0, T_{\max})$. This gives the total mass conservation (9) by

$$\begin{aligned} \frac{d}{dt} \int_\Omega u &= \int_\Omega u_t = \int_\Omega \nabla \cdot (\nabla u - u \nabla v) \\ &= \int_{\partial\Omega} \left( \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) = 0. \end{aligned} \tag{13}$$

A more important feature is the existence of the Lyapunov function

$$W(u,v) = \int_\Omega \left( u \log u - uv + \frac{1}{2} |\nabla v|^2 + \frac{a}{2} v^2 \right).$$

To see this, let us write the first equation of (1) as

$$u_t = \nabla \cdot u \nabla (\log u - v).$$

Then, in use of the boundary conditions we obtain

$$\int_\Omega u_t (\log u - v) = - \int_\Omega u \left| \nabla (\log u - v) \right|^2,$$

where the left-hand side is equal to

$$\frac{d}{dt} \int_\Omega (u \log u - uv) - \int_\Omega u_t + \int_\Omega u v_t.$$

Here, we have (13) and

$$\int_\Omega u v_t = \int_\Omega (\tau v_t - \Delta v + av) v_t = \tau \|v_t\|_2^2 + \frac{1}{2} \frac{d}{dt} \left( \|\nabla v\|_2^2 + a \|v\|_2^2 \right).$$

Therefore,

$$\frac{d}{dt}W(u,v)+\tau\|v_t\|_2^2+\int_\Omega u\left|\nabla\left(\log u-v\right)\right|^2=0 \qquad (t\in[0,T_{\max})) \tag{14}$$

follows. In particular,

$$\frac{d}{dt}W(u,v)\leq 0$$

and we have

$$W\left(u(t),v(t)\right)\leq W\left(u_0,v_0\right) \tag{15}$$

for $t\in[0,T_{\max})$.

In the simplified system, we have

$$v(t)=\left(-\Delta_N+a\right)^{-1}u(t)=\int_\Omega G(\cdot,y)u(y,t)dy$$

and the Lyapunov function $W(u,v)$ is reduced to

$$\mathcal{F}(u)=\int_\Omega u\log u-\frac{1}{2}\int\int_{\Omega\times\Omega}G(x,y)u\otimes udxdy, \tag{16}$$

which is nothing but the free energy described in the previous section. In this way, relations (13) and (15), that is, total mass conservation and decreasing of the free energy are obtained.

The first term of $W(u,v)$, that is $\int_\Omega u\log u$, is related to the Zygmund norm. Actually, the Orlicz space $L\log L(\Omega)$ is provided with the norm

$$[w]_{L\log L}=\int_\Omega|w|\log\left(e+\frac{|w|}{\|w\|_1}\right).$$

See Iwaniec and Verde [13]. We note that $L\log L(\Omega)$ and $\mathrm{Exp}(\Omega)$ form a duality, which is regarded as a local version of that between the Hardy space $\mathcal{H}^1$ and the BMO. We can regard the second term of $W(u,v)$, $\int_\Omega uv$, as a paring of this duality. In fact, the third term of $W(u,v)$, that is $\frac{1}{2}\|\nabla v\|_2^2+\frac{a}{2}\|v\|_2^2$, becomes the square of the $H^1$ norm and we have the inclusion $H^1\subset BMO$ in the case of two space dimensions. Those observations are useful, especially, in the study of stability of staionary solutions to the full system. See [31] and [29].

Relation (14) is also useful to formulate the stationary problem, where $u=u(x)$ and $v=v(x)$ are inedpendent of $t$. Actually, in this case we have

$$\int_\Omega u\left|\nabla\left(\log u-v\right)\right|^2=0.$$

Because we are interested in the non-trivial case $u(x) > 0$, it gives that $\log u - v =$ constant on $\overline{\Omega}$. We prescribe this unknown constant by $\|u\|_1 = \lambda$, taking regards to (9). Consequently, the relation

$$u = \lambda e^v / \int_\Omega e^v$$

is obtained, and thus the stationary problem of (1) arises from the second equation as

$$-\Delta v + av = \lambda e^v / \int_\Omega e^v \quad \text{in} \quad \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{17}$$

where $\lambda = \|u_0\|_1$. This is actually the statinary problem of (1) formulated by Childress and Percus [5].

Problem (17) has several relatives such as the mean field equation of vortex points, the prescribed Gaussian curvature equation on compact Riemannian manifolds, the multi-vortex equation of the Chern-Simons-Higgs gauge theory, and so forth. See [30], [22], and the references therein for details.

Stationary problem (17) has a variational structure. Namely, $v = v(x)$ is a solution if and only if it is a critical point of

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\left(\|\nabla v\|_2^2 + a\|v\|_2^2\right) - \lambda \log\left(\int_\Omega e^v\right) \qquad \left(v \in H^1(\Omega)\right),$$

where the Trudinger-Moser inequality takes a fundamental role. Furthermore, the linearized operator around the stationary solution $v = v(x)$ is associated with the bi-linear form

$$\mathcal{A}(\varphi, \varphi) = \int_\Omega \left(|\nabla\varphi|^2 + a\varphi^2 - p\varphi^2\right) + \frac{1}{\lambda}\left\{\int_\Omega p\varphi\right\}^2 \qquad \left(\varphi \in H^1(\Omega)\right),$$

where $p = \lambda e^v / \int_\Omega e^v$. In this way, methods developed by [30], use of the complex variables, spectral analysis combined with the isoperimetric inequalities on surfaces, control of Palais-Smale sequences by Struwe's argument, and so on, are applicable to (17). See [24] and [22] concerning the structure of the solution set obtained in those ways.

While (17) is the stationary problem described in $v$, that in $u$ is expressed as

$$\log u - A^{-1}u = \text{constant} \qquad \text{and} \qquad \|u\|_1 = \lambda.$$

It is equivalent for $u$ to be a satationary point of $\mathcal{F}(u)$ defined by (16) on $\|u\|_1 = \lambda$. In [29], it is shown that those variational structures are equivalent

up to the Morse indices. Here we just mention key identities for this fact to hold:

$$W\left(\lambda e^v / \int_\Omega e^v, v\right) = \mathcal{J}_\lambda(v) + \lambda \log \lambda \qquad \text{and} \qquad W\left(u, A^{-1}u\right) = \mathcal{F}(u).$$

Simplified system (1) has one more remarkable structure, which may be referred to as the *compensated compactness via the symmetrization.* In fact, taking $\psi \in C^2(\overline{\Omega})$ in $\frac{\partial \psi}{\partial \nu}\Big|_{\partial\Omega} = 0$ as the test function and in use of (4) for the second equation, we get the weak formulation,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \psi(x)u(x,t)dx - \int_\Omega \Delta\psi(x)u(x,t)dx \\
&\quad = \int_\Omega u(x,t)\nabla v(x,t)\cdot\nabla\psi(x)dx \\
&\quad = \int\int_{\Omega\times\Omega} \nabla\psi(x)\cdot\nabla_x G(x,y)u(x,t)u(y,t)dxdy \\
&\quad = \frac{1}{2}\int\int_{\Omega\times\Omega} \rho_\psi(x,y)u(x,t)u(y,t)dxdy,
\end{aligned}$$

where

$$\rho_\psi(x,y) = \nabla\psi(x)\cdot\nabla_x G(x,y) + \nabla\psi(y)\cdot\nabla_y G(x,y).$$

If we apply

$$G(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} + H(x,y)$$

with $H \in C^{1,\theta}(\Omega\times\Omega)$, we know that

$$\rho_\psi(x,y) = -\frac{(\nabla\psi(x)-\nabla\psi(y))\cdot(x-y)}{2\pi|x-y|^2} + C^\theta(\Omega\times\Omega),$$

where the first term of the right-hand side is in $L^\infty$ in $\Omega\times\Omega$ although it is not continuous. More delicate analysis is necessary near $\partial\Omega$, but an important consequence of the above expression is that the local $L^1$ norm of $u$ has a bounded variation in $t \in [0, T_{\max})$. This fact implies the finiteness of blowup points in the simplified system. See [25] for details.

# 4 Parabolic Blowup Point

We come back to the problem of mass quantization, $m(x_0) = m_*(x_0)$ in (7). Let $\varphi \in C_0^\infty(\mathbf{R}^2)$ be in

$$0 \le \varphi \le 1, \qquad \varphi(x) = \begin{cases} 1 & (|x| < 1/2) \\ 0 & (|x| > 1) \end{cases}$$

and set $\psi = \varphi^4$. Given $x_0 \in \mathcal{S}$, we set $\psi_{R,x_0}(x) = \psi\left(\frac{x-x_0}{R}\right)$ and

$$M_{R,x_0}(t) = \int_\Omega \psi_{R,x_0}(x) u(x,t) dx.$$

Then, relation (7) gives that

$$\lim_{R \downarrow 0} \lim_{t \uparrow T_{\max}} M_{R,x_0}(t) = m(x_0).$$

We say that $x_0 \in \mathcal{S}$ is parabolic if

$$\lim_{t \uparrow T_{\max}} M_{R_b(t),x_0}(t) = m(x_0) \tag{18}$$

holds for any $b > 0$ sufficiently small, where $R_b(t) = b\,(T_{\max} - t)^{1/2}$. Under this notation, our theorem is stated as follows.

**Theorem 2** *If $x_0 \in \mathcal{S}$ is parabolic, then it holds that $m(x_0) = m_*(x_0)$.*

Note that $y = (x - x_0)/R_b(t)$ is the standard backward self-similar transformation. It always holds that

$$\limsup_{t \uparrow T_{\max}} M_{R_b(t),x_0}(t) \le m(x_0)$$

and hence (18) is equivalent to

$$\liminf_{t \uparrow T_{\max}} M_{R_b(t),x_0}(t) \ge m(x_0).$$

Relation (18) indicates that the concentration of (7) is enveloped in the parabolic region associated with that transformation. This is not the case for sub-critical nonlinearity as Giga and Kohn [8] shows. In fact, the blowup mechanism of the parabolic equation

$$u_t - \Delta u = u^p, \quad u \ge 0 \qquad \text{in} \qquad \Omega \times (0,T)$$

with $u|_{\partial\Omega} = 0$ is controlled by the ODE part $\dot{u} = u^p$ if the nonlinearity is sub-critical as $p \in (1, \frac{n+2}{n-2})$, where $\Omega \subset \mathbf{R}^n$ is a bounded convex domain. Namely, if $x_0$ is a blowup point, then

$$u(x,t) = (T-t)^{-\frac{1}{p-1}} \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} \{1 + o(1)\}$$

holds as $t \uparrow T = T_{\max}$ uniformly in $|x - x_0| \le C(T-t)^{1/2}$. In this case, the concentration is so slow that $u(x,t)$ becomes flat in any parabolic region. That is, the total blowup mechanism is not enveloped there.

On the other hand, it has been observed that the blowup rate in (1) is super-critical. This will assure the concentration envelope included in the parabolic region. Namely, the concentration must be so rapid as the solution rescalled in the parabolic region will form the collapse again. In fact, the radially symmetric solution constructed by Herrero and Velázquez [12] has the form

$$\begin{aligned} u(x,t) = \frac{1}{r(t)^2}\overline{u}\left(\frac{x}{r(t)}\right)\{1 + o(1)\} \\ + O\left(\frac{e^{-\sqrt{2}|\log(T-t)|^{1/2}}}{|x|^2} \cdot 1_{\{|x| \ge r(t)\}}\right) \end{aligned} \tag{19}$$

as $t \uparrow T = T_{\max}$ uniformly in $|x| \le C(T-t)^{1/2}$, where

$$\begin{aligned} r(t) = C(T-t)^{1/2} \cdot e^{-\sqrt{2}/2\ |\log(T-t)|^{1/2}} \\ \cdot |\log(T-t)|^{\frac{1}{4}\log^{-1/2}(T-t) - \frac{1}{4}} (1 + o(1)) \end{aligned}$$

and $\overline{u}(y) = 8 \cdot \left(1 + |y|^2\right)^{-2}$. We have $0 < r(t) \ll R_b(t)$ and (19) implies (18). In this case the origin is actually a parabolic blowup point.

Now, we shall give the proof of Theorem 2. Let us recall that $\lambda = \|u_0\|_1$. In the following, $C_i$ $(i = 1, 2)$ indicate positive constants determined by $\Omega$. It is known that

$$\left|\frac{d}{dt}\int_\Omega \xi(x)u(x,t)dx\right| \le C_1\left(\lambda + \lambda^2\right)\|\xi\|_{C^2(\overline{\Omega})} \tag{20}$$

holds for $\xi \in C^2(\overline{\Omega})$ in $\frac{\partial\xi}{\partial\nu}\Big|_{\partial\Omega} = 0$. Recall, also, $\psi_{R,x_0}(x) = \psi\left((x - x_0)/R\right)$ for $\psi = \varphi^4$, and introduce the second moment

$$I_{R,x_0}(t) = \int_\Omega |x - x_0|^2 \psi_{R,x_0}(x)u(x,t)dx.$$

Henceforth, we shall write $\psi_R(x) = \psi_{R,x_0}(x)$, $M_R(t) = M_{R,x_0}(t)$, $I_R(t) = I_{R,x_0}(t)$, and $R(t) = R_b(t)$ for simplicity.

Without loss of generality, we take the case $x_0 \in \Omega$. Similarly to Lemma 2.1 of [27], we have for

$$M_R(t) = \int_\Omega \psi_R(x) u(x,t) dx$$

that

$$\frac{dI_R}{dt} \leq 4M_R - \frac{M_R^2}{2\pi} + C_2 R^{-1} \left(\lambda^{3/2} + \lambda^{1/2}\right) I_{3R}^{1/2}.$$

Here, we have

$$\begin{aligned} I_{3R}(t) &= I_R(t) + \int_\Omega |x - x_0|^2 \left(\psi_{3R}(x) - \psi_R(x)\right) u(x,t) dx \\ &\leq I_R(t) + 9R^2 \int_\Omega \left(\psi_{3R}(x) - \psi_R(x)\right) u(x,t) dx \end{aligned}$$

and hence

$$\begin{aligned} \frac{dI_R}{dt} \leq\ & 4M_R - \frac{M_R^2}{2\pi} + C_2 R^{-1} \left(\lambda^{3/2} + \lambda^{1/2}\right) I_R^{1/2} \\ & + 3C_2 \left(\lambda^{3/2} + \lambda^{1/2}\right) \left\{ \int_\Omega \left(\psi_{3R}(x) - \psi_R(x)\right) u(x,t) dx \right\}^{1/2} \end{aligned}$$

follows. We have from (20) that

$$\begin{aligned} \frac{dI_R}{dt} \leq\ & 4M_R(0) - \frac{M_R(0)^2}{2\pi} + C_2 R^{-1} \left(\lambda^{3/2} + \lambda^{1/2}\right) I_R^{1/2} \\ & + 3C_2 \left(\lambda^{2/3} + \lambda^{1/2}\right) \left\{ \int_\Omega \left(\psi_{3R}(x) - \psi_R(x)\right) u_0(x) dx \right\}^{1/2} \\ & + C_3 \left(\lambda + \lambda^{5/2}\right) \left(R^{-2} t + R^{-1} t^{1/2}\right). \end{aligned}$$

We also have

$$\begin{aligned} \int_\Omega \left(\psi_{3R}(x) - \psi_R(x)\right) u_0(x) dx &\leq \int_{B(x_0,3R)\setminus B(x_0,R/2)} u_0(x) dx \\ &\leq 4R^{-2} I_{3R}(0). \end{aligned}$$

Writing $B = C_2 \left(\lambda^{3/2} + \lambda^{1/2}\right)$, $a(s) = C_3 \left(\lambda + \lambda^{5/2}\right)(s^2 + s)$, and

$$J_R(t) = 4M_R(t) - \frac{M_R(t)^2}{2\pi} + 6BR^{-1} I_{3R}(t)^{1/2},$$

we obtain

$$\frac{dI_R}{dt} \leq J_R(0) + a\left(R^{-1}t^{1/2}\right) + BR^{-1}I_R(t)^{1/2}. \tag{21}$$

First, we take the case that $J_R(0) = -A < 0$ and $T \equiv a^{-1}(A/4)^2 \cdot R^2 < T_{\max}$. Then, we have

$$a^{-1}(R^{-1}t^{1/2}) \leq a^{-1}(R^{-1}T^{1/2}) = A/4$$

and hence

$$\frac{dJ_R}{dt} \leq -\frac{A}{4} + BR^{-1}I_R^{1/2}$$

holds for $t \in [0, T]$. Therefore,

$$\frac{1}{R^2}I_R(0) < \left(\frac{A}{24B}\right)^2 \qquad \text{and} \qquad I_R(0) < \frac{A}{6} \cdot T = \frac{R^2}{6}a^{-1}\left(\frac{A}{4}\right)^2$$

imply

$$\left.\frac{dI_R}{dt}\right|_{t=0} \leq -\frac{A}{6}$$

and hence

$$\frac{1}{R^2}I_R(t) < \left(\frac{A}{24B}\right)^2 \qquad \text{and} \qquad \frac{dI_R}{dt} \leq -\frac{A}{6}$$

follow for $t \in [0, T)$. Therefore, we get

$$I_R(t) \leq I_R(0) - \frac{A}{6} \cdot T < 0,$$

a contradiction. In other words,

$$\frac{1}{R^2}I_R(0) \geq \min\left\{\frac{1}{6}a^{-1}\left(\frac{A}{4}\right)^2, \left(\frac{A}{24B}\right)^2\right\}$$

holds in this case.

The other case is indicated as $J_R(0) \geq 0$ or

$$-J_R(0) \geq 4 \cdot a\left(\frac{T_{\max}^{1/2}}{R}\right) \tag{22}$$

In any case, we have we have either (22) or

$$\frac{1}{R^2}I_R(0) \geq \min\left\{\frac{1}{6}a^{-1}\left(\min\left(0, -\frac{J_R(0)}{4}\right)\right), \min\left(0, \frac{-J_R(0)}{24B}\right)^2\right\}.$$

Because system (1) is autonomous in $t$, the following alternatives hold for each $R > 0$ and $t \in [0, T_{\max})$:

$$\text{(i)} \quad -J_R(t) \geq 4 \cdot a\left(\frac{(T_{\max}-t)^{1/2}}{R}\right)$$

$$\text{(ii)} \quad \frac{1}{R^2} I_R(t) \geq \min\left\{6a^{-1}\left(\min\left(0, \frac{-J_R(t)}{4}\right)\right), \min\left(0, \frac{-J_R(t)}{24B}\right)^2\right\}$$

Now, we show the following.

**Lemma 3** *If $x_0 \in \mathcal{S}$ is parabolic, then it holds that*

$$\lim_{t \uparrow T_{\max}} \frac{1}{R(t)^2} I_{R(t)}(t) = 0.$$

*Proof:* From the assumption we have

$$\lim_{t \uparrow T_{\max}} \left\{M_{R(t)}(t) - M_{\varepsilon R(t)}(t)\right\} = 0$$

for any $\varepsilon \in (0,1)$. Here, we have

$$\begin{aligned}
\frac{1}{R(t)^2} I_{R(t)}(t) &= \frac{1}{R(t)^2} \int_\Omega |x - x_0|^2 \psi_{R(t)}(x) u(x,t) dx \\
&= \frac{1}{R(t)^2} \int_\Omega |x - x_0|^2 \left(\psi_{R(t)}(x) - \psi_{\varepsilon R(t)}(x)\right) u(x,t) dx \\
&\quad + \frac{1}{R(t)^2} \int_\Omega |x - x_0|^2 \psi_{\varepsilon R(t)}(x) u(x,t) dx \\
&= \frac{1}{R(t)^2} \int_{|x-x_0| \leq R(t)} |x - x_0|^2 \left(\psi_{R(t)}(x) - \psi_{\varepsilon R(t)}(x)\right) u(x,t) dx \\
&\quad + \frac{1}{R(t)^2} \int_\Omega |x - x_0|^2 \psi_{\varepsilon R(t)}(x) u(x,t) dx \\
&\leq \int_\Omega \left(\psi_{R(t)}(x) - \psi_{\varepsilon R(t)}(x)\right) u(x,t) dx + \varepsilon^2 \lambda \\
&= \left\{M_{R(t)}(t) - M_{\varepsilon R(t)}(t)\right\} + \varepsilon^2 \lambda.
\end{aligned}$$

Making $t \uparrow T_{\max}$ and then $\varepsilon \downarrow 0$, we obtain the conclusion.

Let us complete the proof of Theorem 2. In fact, we have $M_{R(t)}(t) \to m(x_0)$ for $R(t) = b\,(T_{\max} - t)^{1/2}$ and hence

$$J_{R(t)}(t) \quad \to \quad 4m(x_0) - \frac{m(x_0)^2}{2\pi}$$

because

$$\lim_{t \uparrow T_{\max}} \frac{1}{R(t)} I_{3R(t)}(t) = 0$$

holds similarly to Lemma 3. Applying the alternatives (i) and (ii) with $R = R(t)$, we get

$$4m(x_0) - \frac{m(x_0)^2}{2\pi} \begin{cases} \le -4a(b^{-1}) \\ \text{or} \\ \ge 0. \end{cases}$$

The first alternative is impossible if $b > 0$ is small. Therefore, the second alternative follows and hence $m(x_0) \le 8\pi$ is proven.

# 5 Concluding Remarks

Above considerations lead to the idea that the standard rescalling makes the blowup mechanism clearer. In fact, if $T = T_{\max} < +\infty$, $y = x/R_b(t)$, and $e^{-s} = T - t$, then $z(y,s) = (T-t)u(x,t)$ satisfies a similar system to (1). Because $\{z(s)\}$ is a global orbit, we can argue as in [28]. It says that if $u(x,t)$ is a solution to (1) globally in time, then any $t_n \uparrow +\infty$ admits $\{t_n'\} \subset \{t_n\}$ and $0 \le f \in L^1(\Omega)$ such that

$$u(x,t_n')dx \rightharpoonup \sum_{x_0 \in \mathcal{B}(\{t_n'\})} m_*(x_0)\delta_{x_0}(dx) + f(x)dx, \tag{23}$$

where $\mathcal{B}(\{t_n'\})$ denotes the set of *exausted* blowup points so that $x_0$ belongs to it if and only if there is $\{x_n'\} \subset \overline{\Omega}$ such that $u(x_n', t_n') \to +\infty$.

What we conjecture now is that in the rescalled system the same thing occurs with $f = 0$. Coming back to the original system, this implies that $M_{R_b(t)}(t)/m_*$ accumulates to $\{0, 1, \cdots\}$ as $t \uparrow T_{\max}$. However, this can control outside the parabolic region thanks to (20), and $m(x_0)/m_*(x_0) \in \mathcal{N} = \{1, 2, \cdots\}$ follows in (7).

It may not be surprising if the multi-quantization $m(x_0) = n \cdot m_*(x_0)$ occurs with $n = 2, 3, \cdots$ in spite that in the rescalled space-time system they are separated as (23). In other words, only large parabolic region can contain the full blowup mechanism and smaller one may lose multi-collapses. This gives us another conjecture about the concentration speed although details are not described here.

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